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On the propagation of nonlinear acoustic waves in viscous and thermoviscous fluids

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ABSTRACT

Acoustic travelling waves are studied in the context of nonlinear propagation in Newtonian fluids. First, we examine the one-dimensional (1D) versions of four weakly-nonlinear acoustic models, all of which admit known travelling wave solutions (TWS)s in the form of Taylor shocks. Next, we derive the exact, but implicit, kink-type TWS of the compressible 1D Navier–Stokes equations under the homentropic assumption. Then, using three simple metrics, we numerically compare the TWSs of the former with that of the latter. It is shown that, while results for gases are mixed, in the case of liquids, the simple Burgers' equation yields the TWS that best approximates the profile of the Navier–Stokes equations. Lastly, an energy analysis of the weakly-nonlinear models is carried out, connections between the models are noted, and new critical values of the physical parameters are presented.

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1. Introduction

The study of nonlinear acoustic phenomena in dissipative fluids has been, and remains, an important branch of fluid mechanics. In the 1950s, Mendousse [1], and later Lighthill [2], showed that under certain assumptions, the essential one being that fluctuations in the field variables are sufficiently small in amplitude, Burgers' equation provides an approximate description of "finite amplitude" acoustic waves in dissipative perfect gases; see also Ref. [3] and those therein. Since that time researchers have, in an effort to obtain ever more accurate, but still tractable, descriptions of acoustic propagation in gases *and* liquids, derived a number of other so-called weakly-nonlinear model equations, i.e., nonlinear approximations to the compressible Navier–Stokes equations.

The primary aim of the present study is to determine, for physically realistic parameter regimes, which of the following bi-directional, weakly-nonlinear models best approximates the compressible 1D Navier–Stokes equations under the travelling wave assumption.

Lighthill–Westervelt (LW) equation [4,5]:

$$c_e^2\phi_{xx} - \phi_{tt} + \delta c_e^{-2}\phi_{ttt} = \beta c_e^{-2}\partial_t(\phi_t)^2. \tag{1}$$

Blackstock–Lesser–Seebass–Crighton (BLSC) equation [6,7]:

$$[c_e^2 - 2(\beta-1)\phi_t]\phi_{xx} - \phi_{tt} + \delta\phi_{txx} = \partial_t(\phi_x)^2. \tag{2}$$

Kuznetsov's equation [8,9]:

$$c_e^2\phi_{xx} - \phi_{tt} + \delta\phi_{txx} = \partial_t[(\phi_x)^2 + c_e^{-2}(\beta-1)(\phi_t)^2]. \tag{3}$$

Rasmussen–Sørensen–Gaididei–Christiansen (RSGC) equation [10,11]:

$$(c_e^2 - \phi_t)\phi_{xx} - \phi_{tt} + \delta\phi_{txx} = \partial_t\left[(\phi_x)^2 + c_e^{-2}\left(\beta - \frac{3}{2}\right)(\phi_t)^2\right]. \tag{4}$$

These four models are based on the assumption that the propagation medium is a *classical thermoviscous fluid*[5] whose equilibrium (or undisturbed) state is characterised by $\varrho = \varrho_e, \wp =$

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[5] By which we mean a Newtonian fluid whose transport coefficients are constants, and within which the flow of heat is governed by Fourier's law; see, e.g., [12, Chap. 10].

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14. ABSTRACT
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$\wp_e, \mathbf{u} = (0,0,0), \theta = \theta_e$, and $\eta = \eta_e$, where $\varrho, \wp, \mathbf{u}, \theta$ and η denote the mass density, thermodynamic pressure, velocity, absolute temperature, and specific entropy, respectively. Additionally, all material parameters, as well as all quantities with an "e" subscript, are regarded as constant; the velocity vector is taken as $\mathbf{u} = (u(x,t), 0, 0)$, i.e., planar flow along the x-axis is assumed; $u = \phi_x$, where $\phi = \phi(x,t)$ denotes the velocity potential, by virtue of the fact that the irrotationality condition $\nabla \times \mathbf{u} = 0$ is identically satisfied here; and all body forces are neglected. Moreover, $c_e(>0)$, the adiabatic sound speed, denotes the sound speed in the undisturbed fluid; β is known as the *coefficient of nonlinearity* [13], where $\beta \in (1,7)$ for nearly all fluids under ordinary conditions; and

$$\delta = \nu[4/3 + \mu_B/\mu + (\gamma - 1)/\Pr], \tag{5}$$

termed the *diffusivity of sound* by Lighthill [2], is a positive constant [14, Sect. 4.13]. In Eq. (5), the constants $\mu(>0)$ and μ_B respectively denote the shear and bulk viscosities, where we note that μ_B is generally nonzero [14, p. 21]; $\nu = \mu/\varrho_e$ is the kinematic viscosity; the adiabatic index is defined as $\gamma := c_p/c_v$, where the constants $c_p > c_v > 0$ respectively denote the specific heats at constant pressure and volume [14,15]; and $\Pr = \nu/\kappa$ is the Prandtl number, where κ is the thermal diffusivity.

The approach we adopt here is to first obtain, wherever possible, analytical results, i.e., exact and/or approximate solutions, and then turn to numerical methods for the purposes of comparison and illustration. The present study could be regarded as a follow-on to those presented in Refs. [11, Sect. 5.2] and [16], the aims of which were to compare several lossless weakly-nonlinear models with the Euler equations.

In drawing this section to a close, we feel it necessary to point out the following. The four PDEs we consider are all, as a survey of the acoustics literature reveals, well-established, weakly-nonlinear approximations of the 1D Navier–Stokes equations. They have been derived, as alluded to earlier, primarily so that researchers might have simplified (i.e., analytically tractable) alternatives to the latter; in other words, as single-equation models that are, nonetheless, capable of capturing the nonlinear phenomena exhibited by Newtonian fluids undergoing irrotational compressible flow. To be clear, our purpose here is *not* to attempt to improve upon Eqs. (1)–(4) by including higher order nonlinearities; indeed, doing so, assuming all requisite higher-order terms in the constitutive relations are known unambiguously [15, Sect. 1.1.5], is not only counter to the reason(s) these PDEs were originally derived, but negates their usefulness as models by making them more complicated, possibly to the point where the analytical challenges they would then pose rivaled those of the (1D) Navier–Stokes equations themselves.

Remark 1. In the case of perfect gases, $c_e = \sqrt{\gamma \wp_e/\varrho_e}$ and β can be expressed in terms of γ, specifically, by $\beta = (\gamma+1)/2$.

2. Governing equations

All four weakly nonlinear models are based on the continuity and momentum equations, which in 1D reduce to

$$\varrho_t + u\varrho_x + \varrho u_x = 0, \tag{6}$$

$$\varrho(u_t + uu_x) = -\wp_x + \left(\frac{4}{3}\mu + \mu_B\right) u_{xx}, \tag{7}$$

respectively, recalling that all body forces have been neglected, as well as the *linearised* 1D energy equation [17, p. 21]

$$\varrho_e\theta_e\eta_t = K\theta_{xx} = -\varkappa\kappa\varrho_e\theta_e\phi_{txx}, \tag{8}$$

and the assumption that the equation of state can be taken as [15, pp. 581–583]

$$\wp = \wp_e + \varrho_e c_e^2\left[s + (\beta-1)s^2 + \left(\frac{\gamma-1}{\varkappa c_e^2}\right)(\eta - \eta_e)\right]$$
$$(-1 \ll s \ll 1), \tag{9}$$

i.e., as a truncated Taylor expansion of the general constitutive equation $\wp = \wp(\varrho, \eta)$. Here, $s = (\varrho - \varrho_e)/\varrho_e$ denotes the *condensation* and the positive constants K and $\varkappa$ represent the thermal conductivity and the thermal coefficient of volume expansion, respectively, where it should be noted that $K = \kappa c_p \varrho_e$.

Remark 2. As alluded to earlier, since they are based on Eq. (9), Eqs. (1)–(4) describe acoustic waves in both liquids *and* gases, provided of course that fluctuations in ϱ, u, etc., about their equilibrium state values are sufficiently small.

3. TWSs: weakly-nonlinear models

In this section we review and examine exact TWSs of the four weakly-nonlinear models. To this end, we begin by recasting Eqs. (1)–(4) in terms of the following dimensionless quantities:

$$\phi^\diamond = \phi/(UL), \quad u^\diamond = u/U, \quad x^\diamond = x/L, \quad t^\diamond = t(c_e/L). \tag{10}$$

Here, the positive constants U and L denote a characteristic speed and length, respectively, and for typographical convenience all diamond superscripts will henceforth be omitted but are to remain understood. For future reference we note that $\epsilon = U/c_e$ is the Mach number, where $\epsilon \ll 1$ is henceforth assumed; $\mathrm{Re}_d = c_e L/\delta$ denotes a Reynolds number; $\xi(\cdot) := x - (\cdot)t$ is the wave variable; a prime denotes $d/d\xi$; and n is an index which assumes the values $n = 0, 1, 2, 3$.

And lastly, since Eqs. (1)–(4) are all invariant under the transformation $x \mapsto -x$, we henceforth assume, without loss of generality, *only* right-running waves.

3.1. The LW and BLSC equations

In dimensionless form, Eqs. (1) and (2) become, respectively,

$$\phi_{xx} - \phi_{tt} + (\mathrm{Re}_d)^{-1}\phi_{ttt} = \epsilon\beta\partial_t(\phi_t)^2. \tag{11}$$

$$[1 - 2\epsilon(\beta-1)\phi_t]\phi_{xx} - \phi_{tt} + (\mathrm{Re}_d)^{-1}\phi_{txx} = 2\epsilon\phi_x\phi_{xt}. \tag{12}$$

Introducing the ansatzes $u(x,t) = F(\xi(v_n))$ and $\phi(x,t) = \mathcal{P}(\xi(v_n))$, where $F := \mathcal{P}'$, the constants $v_n(>0)$ denote the speeds of the travelling waves, and in this subsection

$$n := \begin{cases} 0, & \text{LW}, \\ 1, & \text{BLSC}, \end{cases} \tag{13}$$

integrating each once w.r.t ξ, and then imposing the asymptotic condition[6] $F \to 0$ as $\xi \to \infty$, Eqs. (11) and (12) reduce to

$$(\mathrm{Re}_d)^{-1}F' - \left(\frac{1 - v_n^2}{v_n^{3-2n}}\right)F = \epsilon\beta F^2 \quad (n = 0, 1), \tag{14}$$

i.e., the *associated* ODEs of the former and latter are Bernoulli equations.

Integrating these ODEs and then imposing the usual wavefront condition $f(0) = 1/2$, the following *Taylor shock* solutions are obtained:

$$F(\xi(v_n)) = \frac{1}{2}\{1 - \tanh[2\xi(v_n)/\ell_n]\} \quad (n = 0, 1). \tag{15}$$

[6] Physically, this implies that the fluid at $\xi = +\infty$ is in its equilibrium state.

Here, the shock thicknesses, ℓ_n, and the v_n are given by

$$\ell_{0,1} = \frac{4}{\epsilon\beta \mathrm{Re_d}}, \qquad v_n = \begin{cases} \dfrac{1 + 2\cos\left[\frac{1}{3}(\theta + 4\pi)\right]}{3\epsilon\beta}, & n = 0, \\ \dfrac{1}{2}\epsilon\beta + \sqrt{1 + \dfrac{1}{4}\epsilon^2\beta^2}, & n = 1, \end{cases} \tag{16}$$

where $\cos(\theta) = (2 - 27\epsilon^2\beta^2)/2$ follows from Cardan's formula [18]. Moreover, the v_n correspond to the smallest, or only,[7] positive root of the equations

$$\epsilon\beta v_n^{3-2n} - v_n^2 + 1 = 0, \tag{17}$$

which arise from the corresponding ODEs in Eq. (14) as a consequence of enforcing the (second) asymptotic condition $F \to 1$ as $\xi \to -\infty$.

It is an interesting fact that, like those of the $n = 2, 3$ cases (see Section 3.2), the wave speed parameter of the $n = 0$ case suffers a bifurcation; specifically, there exists the critical Mach number value

$$\epsilon_0^*(\beta) = \beta_0^*/\beta, \quad \text{where } \beta_0^* := \frac{2}{9}\sqrt{3} \approx 0.385,$$

such that $v_0 \in (1, \sqrt{3}\,]$ for $\epsilon \in (0, \epsilon_0^*]$, but where $v_0 \in \mathbb{C}$ for $\epsilon > \epsilon_0^*$. Here, it is understood that taking $\epsilon = \epsilon_0^*$ is possible only for $\beta \gg \beta_0^*$, since $\epsilon \ll 1$ is required.

It should also be noted that when $\epsilon \in (0, \epsilon_0^*)$, the $n = 0$ case of Eq. (17) admits a second positive root, denoted here by $\overline{V}_0$, where $\overline{V}_0 \in (v_0, \infty)$; however, this root was rejected due to its unstable nature, specifically, the fact that $\overline{V}_0 \to \infty$ as $\epsilon \to 0$.

Finally, we observe that $v_1 > 1$ for common fluids under ordinary conditions, which follows immediately from the fact that

$$v_1 = 1 + \frac{1}{2}\epsilon\beta + \frac{1}{8}\epsilon^2\beta^2 + \cdots \quad (\epsilon \ll 1), \tag{18}$$

and, in turn, that for a given Mach number, $v_0 > v_1$ on $1 < \beta \le \beta_0^*/\epsilon$, the interval over which $v_0 \in \mathbb{R}$.

Remark 3. When the Mach number is expressed in terms of its critical value via the relation $\epsilon = \lambda_0\epsilon_0^*$, where $\lambda_0 \in (0, 1]$, v_0 becomes *independent* of β.

Remark 4. The BLSC equation is easily derived from what has come to be known as the Söderholm equation by simply omitting terms of $\mathcal{O}(\epsilon^2)$ in the latter; see, e.g., Refs. [11,19] and those therein.

3.2. The Kuznetsov and RSGC equations

In dimensionless form, Eqs. (3) and (4), reduce to the indicated special cases of the following "dummy" PDE:

$$\phi_{xx} - \phi_{tt} + (\mathrm{Re_d})^{-1}\phi_{txx} = \epsilon(n-2)\phi_t\phi_{xx} + \epsilon\left[(\phi_x)^2 + \left(\beta - \frac{1}{2}n\right)(\phi_t)^2\right]_t, \tag{19}$$

where

$$n := \begin{cases} 2, & \text{Kuznetsov}, \\ 3, & \text{RSGC}, \end{cases}$$

and we stress that this PDE does *not* have a (known) physical meaning for values of n other than two and three.

Referring the reader to Refs. [9,11] for specifics on the cases $n = 2, 3$, respectively, we observe that under the same ansatzes and conditions introduced in the previous subsection, the associated ODEs of Eq. (19), like those of Eqs. (11) and (12), are of the Bernoulli type, namely,

$$(\mathrm{Re_d})^{-1}F' - \left(\frac{1 - v_n^2}{v_n}\right)F = \epsilon\left[\frac{1}{2}n + \left(\beta - \frac{1}{2}n\right)v_n^2\right]F^2, \tag{20}$$

which when integrated yield the Taylor shock profiles

$$F(\xi(v_n)) = \frac{1}{2}\{1 - \tanh[2\xi(v_n)/\ell_n]\} \quad (n = 2, 3). \tag{21}$$

Here, $v_{2,3}$ denotes the smallest, or only, positive root of the corresponding equations $\Pi_n(v_n) = 0$, where we have defined

$$\Pi_n(v_n) := \epsilon\left(\beta - \frac{1}{2}n\right)v_n^3 - v_n^2 + \frac{1}{2}\epsilon n v_n + 1 \quad (n = 2, 3); \tag{22}$$

the shock thicknesses are given by

$$\ell_{2,3} = \frac{4}{\mathrm{Re_d}\left[\epsilon\left(\beta - \frac{1}{2}n\right)v_n^2 + \frac{1}{2}\epsilon n\right]} = \frac{4v_{2,3}}{\mathrm{Re_d}(v_{2,3}^2 - 1)} \quad (n = 2, 3), \tag{23}$$

where we observe that $v_{2,3} > 1$; and the restriction $\beta \neq 3/2$ is required for the case $n = 3$.

For $\beta \in (1, 3/2)$, a range which includes the β-values of common (e.g., diatomic and monatomic) gases under ordinary conditions, it is not difficult to establish that $v_3 \in (1, 2)$ and that this, the *only* positive root of $\Pi_3 = 0$ when $\beta \in (1, 3/2)$, is of multiplicity one.

If, on the other hand, we assume $\beta \gg \beta_{2,3}^*$, where

$$\beta_2^* := \frac{32}{27} \approx 1.185 \quad \text{and} \quad \beta_3^* := \frac{27 + 7\sqrt{21}}{36} \approx 1.641,$$

and set $\epsilon = \epsilon_n^*(\beta)$, where

$$\epsilon_n^*(\beta) = \begin{cases} \sqrt{\dfrac{\sqrt{\beta}(9\beta - 8)^{3/2} - (27\beta^2 - 36\beta + 8)}{8(\beta - 1)}}, & n = 2, \\ \sqrt{\dfrac{9\sqrt{\beta(\beta - 4/3)^3} - (9\beta^2 - 18\beta + 6)}{9(\beta - 3/2)}}, & n = 3, \end{cases} \tag{24}$$

then it can be shown that

$$\max(v_n) = \begin{cases} \dfrac{1 + \sqrt{1 - 3(\beta - 1)[\epsilon_2^*(\beta)]^2}}{3(\beta - 1)\epsilon_2^*(\beta)}, & n = 2 \\ \dfrac{2 + \sqrt{4 - 18(\beta - 3/2)[\epsilon_3^*(\beta)]^2}}{6(\beta - 3/2)\epsilon_3^*(\beta)}, & n = 3 \end{cases} \quad (\beta \gg \beta_n^*), \tag{25}$$

each of which is a root of multiplicity *two*. Here, we have imposed large-β requirements to ensure $\epsilon_{2,3}^* \ll 1$, as the weakly-nonlinear approximation demands, just as we did in Section 3.1 to ensure $\epsilon_0^* \ll 1$.

When $\beta > 1, 3/2$ and $\epsilon \in (0, \epsilon_{2,3}^*)$, the equations $\Pi_{2,3} = 0$ admit *two* distinct positive roots, the larger of which in each case, denoted here as $\overline{V}_{2,3}$, we reject since they blow-up as $\epsilon \to 0$. Moreover, $\epsilon_{2,3}^*$ are Mach number values at which the corresponding wave speeds exhibit a *bifurcation*; see Ref. [9, Fig. 1][8] for the case $n = 2$. It is also noteworthy that

$$\sqrt{3} < \max(v_2) < \max(v_3) \quad (\beta > 3/2),$$

where $\max(v_{2,3}) \to \sqrt{3}$ as $\beta \to \infty$; i.e., $\max(v_{2,3}) \to \max(v_0)$ as $\beta \to \infty$.

[7] In this article, "only positive root" also refers to a single positive root of multiplicity two.

[8] The quantities $V_{0,1,2}$, ϵ_c, and β_1 that appear in Ref. [9] correspond to $\max(v_2)$, v_2, $\overline{V}_2$, ϵ_2^*, and β_2^*, respectively, of the present article.

Remark 5. On replacing the "small" terms ϕ_{txx} and $(\phi_x)^2$ in the $n = 2$ case of Eq. (19) with ϕ_{ttt} and $(\phi_t)^2$, respectively, based on the $\mathcal{O}(1)$ approximations $\phi_{xx} \simeq \phi_{tt}$ and $\phi_x \simeq -\phi_t$ [7], one obtains Eq. (11), i.e., the LW equation.

Remark 6. Unlike Kuznetsov's equation, the RSGC equation admits a *Hamiltonian* structure in the lossless (i.e., $\mathrm{Re_d} \to \infty$) limit, as do the Euler equations to which this limiting case corresponds; see Ref. [11, Sects. 2.3, 2.4], wherein it is also shown how the BLSC and Kuznetsov equations are derivable from the RSGC equation.

3.3. Energy-rate results

Seeking further insight into the physics captured by these weakly-nonlinear models, we now perform an energy analysis [20] based on their TWSs. To this end, we return to Eqs. (11), (12) and (19), multiply each by ϕ_t, and then integrate over the real line w.r.t x. Thereupon, after integrating by parts, simplifying, and grouping terms, we are led to consider

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \begin{cases} -\varPhi - \dfrac{2}{3}\epsilon\beta\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{-\infty}^{+\infty}(\phi_t)^3\,\mathrm{d}x, & n = 0,\\ -\varPhi - 2\epsilon\left(\beta - \dfrac{3}{2}\right)\displaystyle\int_{-\infty}^{+\infty}(\phi_t)^2\phi_{xx}\,\mathrm{d}x, & n = 1,\\ -\varPhi + \epsilon\displaystyle\int_{-\infty}^{+\infty}(\phi_t)^2\phi_{xx}\,\mathrm{d}x & \\ \qquad - \dfrac{2}{3}\epsilon(\beta - 1)\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{-\infty}^{+\infty}(\phi_t)^3\,\mathrm{d}x, & n = 2,\\ -\varPhi - \dfrac{2}{3}\epsilon\left(\beta - \dfrac{3}{2}\right)\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{-\infty}^{+\infty}(\phi_t)^3\,\mathrm{d}x, & n = 3, \end{cases} \tag{26}$$

which we should point out are all *exact* relations in the sense that, apart from those used in the original derivations of Eqs. (1)–(4), no other approximations have been employed. Here, $\lim_{x\to\pm\infty}\phi_x$, $\lim_{x\to\pm\infty}\phi_t$, and $\lim_{x\to\pm\infty}\phi_{tx}$ were evaluated using the fact that the Taylor shocks given in Sections 3.1 and 3.2 imply $\phi_x = F(x - v_n t)$ and $\phi_t = -v_n F(x - v_n t)$, for $n = 0, 1, 2, 3$; the energy functional, E, is defined as

$$E := E_{\mathrm{R}} + \frac{1}{2}\int_{-\infty}^{+\infty}[(\phi_x)^2 + (\phi_t)^2]\,\mathrm{d}x - t\begin{cases} v_0, & n = 0,\\ v_n(1 + \epsilon v_n), & n = 1, 2, 3, \end{cases} \tag{27}$$

where E_{R} is a constant; and $\varPhi$ denotes the usual dissipation integral, namely,

$$\varPhi := \frac{1}{\mathrm{Re_d}}\begin{cases} \displaystyle\int_{-\infty}^{+\infty}|\phi_t|\phi_{ttt}\,\mathrm{d}x, & n = 0,\\ \displaystyle\int_{-\infty}^{+\infty}(\phi_{xt})^2\,\mathrm{d}x, & n = 1, 2, 3. \end{cases} \tag{28}$$

Because their integrands involve (at most) products of hyperbolic functions, the integrals on the RHS of Eq. (26) can be evaluated in closed form. Omitting the details, it is readily established, using, e.g., one of the many commercially available symbolic software packages, that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}\epsilon\begin{cases} \beta v_0^4, & n = 0,\\ (\beta - 2)v_1^2, & n = 1,\\ [(\beta - 1)v_2^2 - 1]v_2^2, & n = 2,\\ \left[(\beta - 3/2)v_3^2 - \dfrac{1}{2}\right]v_3^2, & n = 3, \end{cases} \tag{29}$$

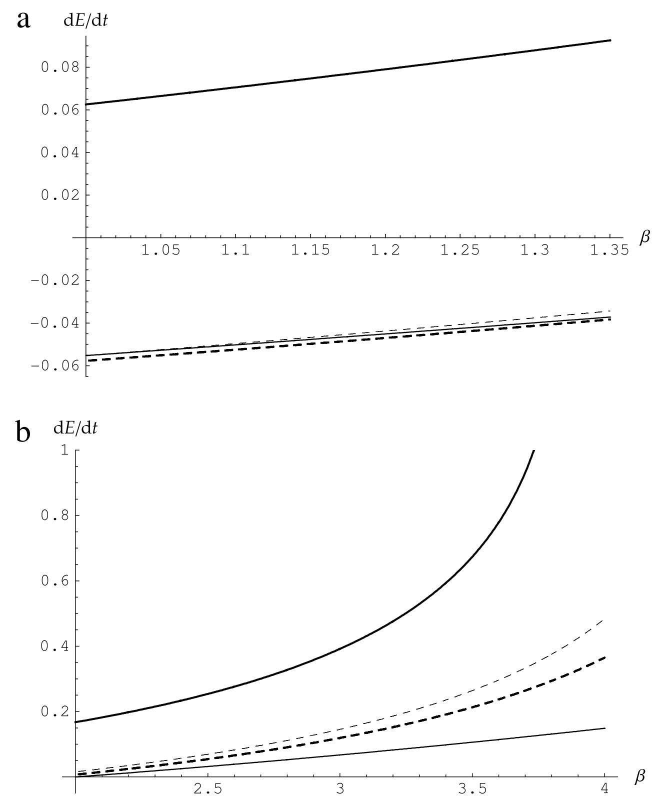

Fig. 1. (a,b) dE/dt vs. β in the case of gases and liquids, respectively, for $\epsilon = 0.1$. Thick-solid: LW ($n = 0$). Thin-solid: BLSC ($n = 1$). Thin-broken: Kuznetsov ($n = 2$). Thick-broken: RSGC ($n = 3$).

which we observe is *independent* of the Reynolds number.[9] We also observe that for $n = 1, 2, 3$, $\mathrm{d}E/\mathrm{d}t \leqslant 0$ in the case of gases (i.e., $1 < \beta < 1.35$) and liquids (i.e., $\beta > 2$), respectively, while $\mathrm{d}E/\mathrm{d}t > 0$ for both gases and liquids when $n = 0$.

To illustrate these results, we have, in Fig. 1, plotted the expressions for dE/dt given in Eq. (29) as functions of β, but with the Mach number fixed at $\epsilon = 0.1$. Therein, we observe that, for both gases and liquids, the curves for the $n = 1, 2, 3$ cases are not only similar in appearance and slope but are also grouped rather closely together, with all appearing below (resp. above) the β-axis in the case of gases (resp. liquids). In contrast, the $n = 0$ curve, which is defined (i.e., real-valued) here in the case of liquids only for $2 < \beta \le \epsilon^{-1}\beta_0^* \approx 3.849$, clearly is removed from the others, always lying above the β-axis and always exhibiting a positive slope.

Remark 7. In obtaining the expressions given in Eq. (29), we found that, on evaluation, the dissipation integral could be expressed as

$$\varPhi = \frac{1}{6}\begin{cases} \epsilon\beta v_n^{4-2n}, & n = 0, 1,\\ v_n(v_n^2 - 1), & n = 2, 3, \end{cases} \tag{30}$$

where, as expected based on Eq. (29), $\varPhi$ in each case turns out to be independent of $\mathrm{Re_d}$. It is also noteworthy that, under each of the weakly-nonlinear models, $\varPhi$ is both strictly positive, as the physics of thermoviscous flow demands, and an increasing polynomial function of the v_n.

[9] Meaning that dE/dt is also independent of δ; see Ref. [12, p. 616], wherein mention of this point is made in relation to Burgers' equation.

4. TWS: 1D Navier–Stokes equations

Our aim in this section is to determine, by analytical methods, a standard against which the TWSs presented in Section 3 can be meaningfully compared/contrasted; specifically, to solve Eqs. (6)–(9) under the traveling wave assumption. While this task might appear all but impossible, given the number of equations involved and their mathematical complexity, a kink-type TWS of this system can, nevertheless, be obtained by regarding the flow as *homentropic* [14,21], which refers to the $\eta = \eta_e$ special case of *isentropic* flow, as demonstrated recently by Rasmussen [11, Eq. (3.41)].

4.1. Ansatzes, associated ODE, and stability results

We now proceed under the homentropic flow approximation,[10] which translates here into letting $K \to 0$ and then integrating equation (8) subject to $\eta(x,0) = \eta_e$. To this end, we return to Eqs. (6), (7) and (9) and use the third, which now assumes *barotropic* [14] form, to eliminate $\wp$ from the second. After recasting the resulting equations in dimensionless form and setting $\nu_L := \nu(4/3 + \mu_B/\mu)$, we end up with the coupled pair

$$s_t + \epsilon u s_x + \epsilon u_x(1+s) = 0, \tag{31}$$

$$\epsilon(1+s)(u_t + \epsilon u u_x) = -[s + (\beta-1)s^2]_x + \epsilon(\mathrm{Re})^{-1}u_{xx}, \tag{32}$$

where Re is a Reynolds number based on ν_L. Here, in the spirit of the terminology of Hayes [22, p. 38], we call $\nu_L \in [\nu, \delta)$ the *kinematic* longitudinal coefficient of viscosity; it represents the purely viscous contribution to the diffusivity of sound, where we observe that $\delta \to \nu_L$ (from above) as $K \to 0$.

Again seeking right-running waves, we set $u(x,t) = f(\xi)$ and $s(x,t) = g(\xi)$, where $\xi(c) := x - ct$ and c is a positive constant, and then substitute these ansatzes into Eqs. (31) and (32). On integrating the former with respect to ξ, and then using the fact that $u = s = 0$ in the equilibrium state to solve for the resulting constant of integration, our system of PDEs is reduced to

$$g = \epsilon f(c - \epsilon f)^{-1} \quad (f \neq f_s), \tag{33}$$

$$-\epsilon(1+g)f'(c-\epsilon f) = -[g + (\beta-1)g^2]' + \epsilon(\mathrm{Re})^{-1}f'', \tag{34}$$

where $f_s := c\epsilon^{-1}$ and we recall that a prime denotes $\mathrm{d}/\mathrm{d}\xi$.

Next, eliminating g between the last two equations, integrating the result once, and then enforcing the first of our (two) asymptotic conditions, namely, $f \to 0$ as $\xi \to \infty$, the associated ODE for f turns out to be the following special case of *Darboux's* equation [23]:

$$(\mathrm{Re})^{-1}f' = \frac{\epsilon(\beta-1)f^2 + f(c-\epsilon f) - cf(c-\epsilon f)^2}{(c-\epsilon f)^2}, \tag{35}$$

where the ensuing (i.e., second) constant of integration is necessarily zero. Here, we observe that this, rather unusual ODE admits the three equilibrium solutions

$$f_2 = 0, \qquad f_{1,3} = \frac{\beta + 2(c^2-1) \mp \sqrt{Q(c)}}{2\epsilon c}, \tag{36}$$

where we have set $Q(c) := \beta^2 + 4(\beta-1)(c^2-1)$ for convenience, and it should be noted that $Q(c)$ is strictly positive. A phase-plane analysis reveals that $\bar{f} = f_{1,2,3}$ are unstable, stable, and stable, respectively, for all physically allowable values of ϵ and β, where we observe that $f_2 < f_1 < f_s < f_3$.

[10] Of course, only *lossless* fluids, i.e., fluids which are neither viscous nor thermally conducting, are capable of exhibiting homentropic flow; however, as we will see, the impact on the weakly-nonlinear models of assuming $\eta = \eta_e$ is simply a decrease in the value of δ.

Remark 8. From Eq. (5) it is clear that the approximation $\delta \approx \nu_L$ applies only to fluids for which $\mathrm{Pr} \gg \gamma - 1$ is satisfied, examples of which include glycerin [14, Table 4.6], pure water [15, p. 583], and oils [24, p. 80].

4.2. Derivation and analysis of wave speed expressions

Enforcing now the second asymptotic condition, i.e., $f \to 1$ as $\xi \to -\infty$, we find that the speed of our travelling wave profile corresponds to the positive root of $\Lambda(c) = 0$, where we have defined

$$\Lambda(c) := c^3 - 2\epsilon c^2 - (1-\epsilon^2)c - \epsilon(\beta-2), \tag{37}$$

that renders $f_1 = 1$. With regards to the zeros of Eq. (37) we note the following: (a) for $\beta > 2$ (i.e., liquids), $\Lambda(c) = 0$ has exactly one positive root, the value of which is always greater than one; (b) for $\beta < 2$ (i.e., gases), this cubic equation has two *unequal* positive roots, at least one of which lies on the interval $(0,1)$; and (c) because it only arises in the context of multiphase flows (specifically, those involving certain dusty gases [15, p. 581]), which of course are *not* described by the Navier–Stokes equations, the case $\beta = 2$, which would result in a single positive root of multiplicity *two*, will *not* be considered here.

Since we are only interested in kink-type solutions of Eq. (35), we take $c = c_m$, where $f_1 = 1$ *only* when $c = c_m$, and, of course, require $f(0) \in (0,1)$, i.e., $f(0) \in (f_2, f_1)$. Here, using Cardan's formula [18] once again, c_m, the *largest* (or *only*) positive root of $\Lambda(c) = 0$, is found to be

$$c_m = \begin{cases} \dfrac{2}{3}\left[\epsilon + \cos\left(\dfrac{1}{3}\vartheta\right)\sqrt{3+\epsilon^2}\right], & \Delta < 0; \\[2ex] \dfrac{2\epsilon}{3} + \dfrac{3+\epsilon^2}{9\left[(27\epsilon(\beta-4/3)-2\epsilon^3)/54 + \sqrt{\Delta}\right]^{1/3}} \\ \qquad + \left[(27\epsilon(\beta-4/3)-2\epsilon^3)/54 + \sqrt{\Delta}\right]^{1/3}, & \Delta \geq 0; \end{cases} \tag{38}$$

where $\Delta = \Delta(\epsilon, \beta)$, the discriminant of Eq. (37), is

$$\Delta(\epsilon,\beta) = \frac{-4(\beta-1)\epsilon^4 + (27\beta^2 - 72\beta + 44)\epsilon^2 - 4}{108}; \tag{39}$$

and the angle ϑ is defined by the relation

$$\cos(\vartheta) = \frac{\epsilon}{2}\left[\frac{27(\beta-4/3) - 2\epsilon^2}{\sqrt{(3+\epsilon^2)^3}}\right]. \tag{40}$$

Remark 9. An inspection of Eq. (40) revels that $\beta^\bullet := \frac{2}{3}(2+\epsilon^2/9)$ is a critical value in the sense that $\vartheta = \pi/2$ when $\beta = \beta^\bullet$, where we observe that $\Delta(\epsilon, \beta^\bullet) < 0$ and

$$c_m|_{\beta=\beta^\bullet} = \frac{2}{3}\left[\epsilon + \cos\left(\frac{\pi}{6}\right)\sqrt{3+\epsilon^2}\right]. \tag{41}$$

More interesting, however, is the following: Since $\epsilon \ll 1$ is assumed, $\beta^\bullet$ is very close to, but strictly *greater* than, 4/3, the theoretical value of β for monatomic gases under standard conditions; see, e.g., Ref. [14, p. 80], recalling the relation $\beta = (\gamma+1)/2$.

Remark 10. For $\beta > 2$, it can be shown that $\Delta(\epsilon,\beta) \leq 0$ for $\epsilon \leq \tilde{\epsilon}_0$, where $\tilde{\epsilon}_0 < 1$ is given by

$$\tilde{\epsilon}_0 := \sqrt{\frac{27\beta^2 - 72\beta + 44 - \sqrt{(\beta-2)(9\beta-10)^3}}{8(\beta-1)}}, \tag{42}$$

while $\Delta(\tilde{\epsilon}_0, \beta) > 0$ for $\epsilon \in (\tilde{\epsilon}_0, 1)$. However, in spite of the fact that $\Delta(\tilde{\epsilon}_0, \beta) = 0$, the wave speed c_m, unlike its counterparts

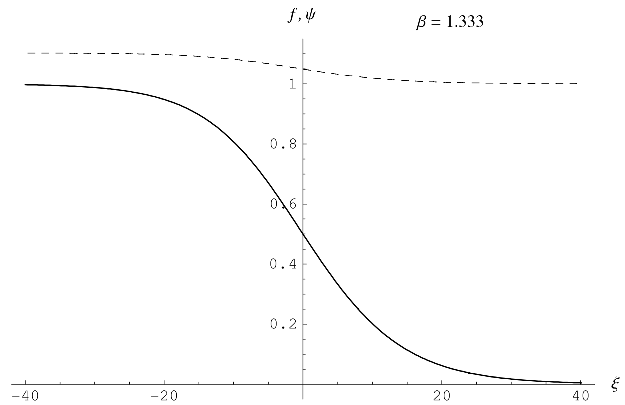

Fig. 2. f, ψ vs. ξ for $\beta = 4/3$ and $\epsilon = 0.1$, which implies $c_m \approx 1.068$, and Re $= 1$. Solid: f vs. ξ (Eq. (43)). Broken: ψ vs. ξ (Eq. (45)).

corresponding to the LW, Kuznetsov, and RSGC equations, *never* suffers a bifurcation. This is easily established by noting that $\Delta < 0$, for every $\epsilon > 0$, when $\beta \in (1, 2)$ and recalling that Eq. (37) admits only one positive root when $\beta > 2$. It is also noteworthy, again assuming $\beta \in (2, 7)$, that $\Delta(1/2, \tilde{\beta}) = 0$, where $\tilde{\beta} := \frac{1}{54}(73 + 13\sqrt{13}) \approx 2.220$.

Remark 11. Since $\epsilon \ll 1$ implies $\epsilon < 1/2$, it follows that $c_m \in (1, 2)$ for every $\beta \in (1, 2) \cup (2, 7)$; i.e., like those of the four weakly-nonlinear models, the travelling wave profiles given below in Eqs. (43) and (45) propagate at *supersonic* speed.

4.3. Exact TWSs for the velocity and density fields

Returning now to Eq. (35), separating variables, and then integrating using partial fractions, we obtain, after enforcing the wavefront condition $f(0) = 1/2$, the following exact, but implicit, TWS for the velocity field (see also Ref. [11, Eq. (3.41)]):

$$\xi(c_m) = \frac{1}{\mathrm{Re}(c_m^3 - c_m)} \left\{ \frac{\beta - 2 - 2c_m^2(\beta - 1)}{\sqrt{Q(c_m)}} \times \tanh^{-1}\left[\frac{\beta - 2 + 2c_m^2(1 - \mathfrak{f})}{\sqrt{Q(c_m)}}\right] - \ln\left[\frac{\mathfrak{f}^{c_m^2}}{\sqrt{c_m^2\mathfrak{f}^2 - \mathfrak{f}(2c_m^2 + \beta - 2) + c_m^2 - 1}}\right]\right\}\Bigg|_{\frac{1}{2}f_m^{-1}}^{f/f_m}, \quad f \in (0, 1). \tag{43}$$

Here, we have set $f_m := c_m/\epsilon$ for convenience and we note that the shock thickness, l, of the kinks (i.e., integral curves of Eq. (35)) described by Eq. (43) is given by

$$l = \frac{2\epsilon(1 - 2f_m)^2}{\mathrm{Re}\,|2\beta - \epsilon c_m(1 - 2f_m)^2 + 4(f_m - 1)|}. \tag{44}$$

In turn, using Eqs. (33) and (43) it is easily shown that ψ, where we now let $\psi(\xi) := 1 + g(\xi)$ denote the dimensionless density, is given in terms of f by

$$\psi(\xi) = 1 + \frac{\epsilon f(\xi)}{c_m - \epsilon f(\xi)} = \frac{f_m}{f_m - f(\xi)}. \tag{45}$$

In Fig. 2 we have, under the assumption that the propagation medium is a monatomic gas, plotted f, ψ vs. ξ for Re $= 1$, which we take without loss of generality,[11] and $\epsilon = 0.1$. As one can clearly see, like that of the velocity field, the density profile assumes the form of a kink, the shock thickness (h) of which is easily determined using Eq. (45):

$$h = \frac{l(1 - 2f_m)^2}{4f_m(f_m - 1)}. \tag{46}$$

Fig. 2 also illustrates the fact that

$$0 < f < 1 < \psi < f_m(f_m - 1)^{-1} \quad (\forall \xi \in \mathbb{R}), \tag{47}$$

where, in Fig. 2, $f_m \approx 10.683$ and $f_m(f_m - 1)^{-1} \approx 1.103$.

Remark 12. Since $f_m > 2$, as can be shown using Eq. (47) and Fig. 2, it follows from Eq. (46) that $h > l$; in other words, viscous dissipation evidently has a greater effect on the density than it does on the velocity field.

[11] Because from the standpoint of numerics, setting Re $= 1$ is equivalent to applying the re-scaling $\xi \mapsto \xi/\mathrm{Re}$.

Table 1
$\beta = 1.047$, $c_m \approx 1.054$, $l \approx 36.319$.

Model	$\mathfrak{D}(F\|n)$	$\mathrm{d}(v_n)$	$\eth(\ell_n)$
LW ($n = 0$)	0.0798	0.00691	1.8857
BLSC ($n = 1$)	0.0798	0.00001	1.8857
Kuz. ($n = 2$)	0.0734	0.00029	1.6964
RSGC ($n = 3$)	0.1436	0.00249	3.7050

Table 2
$\beta = \beta^\bullet$, $c_m \approx 1.068$, $l \approx 28.479$.

Model	$\mathfrak{D}(F\|n)$	$\mathrm{d}(v_n)$	$\eth(\ell_n)$
LW ($n = 0$)	0.0725	0.01265	1.5039
BLSC ($n = 1$)	0.0725	0.00059	1.5039
Kuz. ($n = 2$)	0.0376	0.00324	0.4306
RSGC ($n = 3$)	0.0931	0.00064	2.0351

Table 3
$\beta = 2.350$, $c_m \approx 1.116$, $l \approx 17.000$.

Model	$\mathfrak{D}(F\|n)$	$\mathrm{d}(v_n)$	$\eth(\ell_n)$
LW ($n = 0$)	0.0297	0.05987	0.0215
BLSC ($n = 1$)	0.0297	0.00881	0.0215
Kuz. ($n = 2$)	0.1556	0.03312	2.6180
RSGC ($n = 3$)	0.0978	0.02295	1.6257

5. Numerically evaluated metrics

In this section we compare/contrast the four weakly nonlinear solutions with/to the exact result given in Eq. (43) based on the following three metrics:

$$\mathfrak{D}(F|n) := \sqrt{\int_{-10l}^{10l} [f(\xi) - F(\xi)]^2\, \mathrm{d}\xi}, \qquad \mathrm{d}(v_n) := |c_m - v_n|, \quad \text{and} \quad \eth(\ell_n) := |l - \ell_n|. \tag{48}$$

Here, $\mathfrak{D}(F|n)$ denotes the distance w.r.t the L^2 norm between the kink solution profile given in Eq. (43) and the Taylor shock profile of the nth weakly-nonlinear model, where $\mathrm{Re}_d = \mathrm{Re}$ is to be henceforth understood.

In Tables 1–3, we have computed these three quantities for four different values of β. In every table, as well as in Figs. 3 and 4 which appear in the next section, we have taken $\epsilon = 0.1$ and Re $= 1$. In Table 1, the fluid considered is Butane (C_4H_{10}), in its gas/vapor state, at 20 °C and 1 atm [25]; next, in Table 2, $\beta = \beta^\bullet(\approx 1.334)$ *approximately* corresponds to a monatomic gas; and lastly, in Table 3, $\beta = 2.350$ corresponds to liquid sodium at 110 °C and 1 atm [13].

Considering gases and liquids separately, we seek to numerically identify the weakly-nonlinear model, in the case of each fluid type, that yields the *smallest* value of two or more of the metrics.

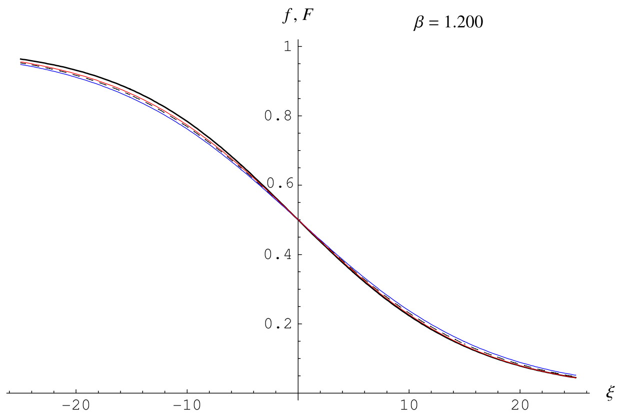

Fig. 3. f, F vs. ξ for $\beta = 1.200$ and $\epsilon = 0.1$, which implies $c_m \approx 1.062$, and Re = 1. (Color available online only.) Black: Navier–Stokes (Eq. (43)). Broken: LW/BLSC (Eq. (15)). Red: Kuznetsov (Eq. (21) with $n = 2$). Blue: RSGC (Eq. (21) with $n = 3$).

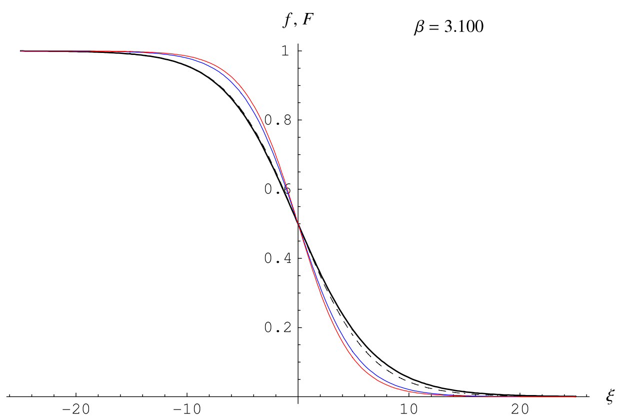

Fig. 4. f, F vs. ξ for $\beta = 3.100$ and $\epsilon = 0.1$, which implies $c_m \approx 1.167$, and Re = 1 (Color available online only.). Black: Navier–Stokes (Eq. (43)). Broken: LW/BLSC (Eq. (15)). Red: Kuznetsov (Eq. (21) with $n = 2$). Blue: RSGC (Eq. (21) with $n = 3$).

6. Closure

6.1. Conclusions

We have, in the travelling wave context, compared/contrasted four bi-directional, weakly-nonlinear acoustic models with/to the 1D compressible Navier–Stokes equations under the assumption of homentropic flow. We have presented exact TWSs for both the former and latter, noted critical values of the parameters, and examined special cases. Based on an analysis of these findings, in particular, the numerical results that have been presented, we report the following:

(i) For $\beta \in (1, \beta^{\bullet})$, i.e., in the case of gases, there is not a unanimous "best approximation" based on our metrics. To be more specific, while Kuznetsov's equation ($n = 2$) yields the smallest values of $\mathfrak{D}(F|n)$ and $\mathfrak{d}(\ell_n)$, the BLSC equation ($n = 1$) gives the smallest value of $\mathsf{d}(v_n)$; see Fig. 3, wherein the value of β taken corresponds to a diatomic gas (e.g., air at 20 °C and 1 atm) [13], along with Tables 1 and 2.

(ii) For $\beta > 2$ (i.e., for liquids), the Burgers' TWS was found to be equal to (in terms of $\mathfrak{D}$ and $\mathfrak{d}$) or better than (in terms of d) that of the BLSC equation, the best of the bi-directional models in the case of liquids; see Fig. 4, wherein the value of β taken corresponds to distilled water at 0 °C [13], as well as Tables 3 and A.1.

(iii) As compared with the other three bi-directional models, the $n = 2$ (resp. $n = 1$) case yields the *smallest* value of $|\mathrm{d}E/\mathrm{d}t|$ for every β in the range corresponding to gases (resp. liquids); see Fig. 1.

(iv) The LW equation had the poorest performance, overall, in terms of wave speed, with the value of $\mathsf{d}(v_0)$ always exceeding not only those of the $n = 1, 2, 3$ cases but also that of Burgers' equation (see Table A.1).

6.2. Discussion

Of our findings, (ii) is clearly the most surprising, given that Burgers' equation (see Appendix A) is actually an evolution equation based on the assumption of plane unidirectional flow. A key aspect of its success, as shown in Table A.1, is the fact that $\mathsf{d}(v_{\mathrm{Beq}}) < \mathsf{d}(v_1)$ when $\beta > 2$, where it is noteworthy that $v_1 = v_{\mathrm{Beq}} + \mathcal{O}(\epsilon^2) > v_{\mathrm{Beq}}$. Of course, if the medium in question is a liquid and bi-directional propagation is a possibility (as in, e.g., acoustic reflection problems), then one would likely be compelled to set aside Burgers' equation in favor of the BLSC equation, which in the context of TWSs is the simplest of the bi-directional models.

It was also something of a surprise, given the results of Refs. [11,16], and the fact that the derivation of the BLSC model involves *fewer* expansions than that of Kuznetsov's equation, to find that the latter outperformed the former, w.r.t two of our three metrics, when only gases were considered; see (i). However, one finds very little difference in both the solution profiles (see Table 1) and energy-rate curves (see Fig. 1(a)) of these two PDEs over $1 < \beta \lesssim 1.1$, a range that includes the β-value of not only Butane but other important gases as well; see, e.g., Refs. [14, p. 640] and [26, p. 788], once again recalling the relation $\beta = (\gamma + 1)/2$.

What is more, the fact that the same bi-directional models that performed best in terms of the metrics $\mathfrak{D}$ and $\mathfrak{d}$ also yielded the smallest values of $|\mathrm{d}E/\mathrm{d}t|$ (see (iii)) suggests that the minimization of $|\mathrm{d}E/\mathrm{d}t|$, over the β-intervals corresponding to gases and liquids, might be the only criteria needed to identify/derive, from among such models, those that best approximate f and l, but not necessarily c_m.

And while the LW equation exhibited a rather poor performance here (see (iv)), as did its lossless version in tests performed by Christov et al. [16], it should be noted that this PDE remains a topic of interest to researchers in a number of acoustics-related fields; see, e.g., Refs. [27–29] and those therein.

6.3. Possible future work

We now mention just a few of the ways in which the present study might be extended. Perhaps the most obvious follow-on investigation would be to carry out a travelling wave analysis using the un-approximated (i.e., nonlinear) 1D energy equation [30, Eq. (14.5.16)] and the non-isentropic equation of state for a perfect gas [5, Eq. (5)] in place of Eqs. (8) and (9), respectively; see also Ref. [31]. Another possibility is to take the constitutive relation for the shear stress as the Ostwald–de Waele power-law [24, p. 25], a formulation that would likely allow for TWSs in the form of *compact* kinks [32]. And lastly, following Straughan [33], one could assume that the flow of heat in the fluid is described not by Fourier's law but instead by Christov's [34] refinement of the Maxwell–Cattaneo law, a result of which would be the introduction of the fluid's thermal relaxation time as a new parameter; see also Ref. [35] and those therein.

Acknowledgments

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Table A.1
For Eq. (A.2) with $\epsilon = 0.1$ and $\mathrm{Re_d} = \mathrm{Re} = 1$.

β	v_{Beq}	$\mathfrak{D}(\mathfrak{F}\|\cdot)$	$\mathrm{d}(v_{\mathrm{Beq}})$	$\mathfrak{d}(\ell_{\mathrm{Beq}})$
1.047	1.052	0.0798	0.00136	1.8857
1.200	1.060	0.0773	0.00158	1.7178
$\beta^{\bullet}$	1.067	0.0725	0.00163	1.5039
2.350	1.118	0.0297	0.00193	0.0215
3.100	1.155	0.0477	0.00814	0.6203

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Appendix. Burgers' equation

As shown by Crighton [7], under the assumption of unidirectional flow, Burgers' equation can be derived from the BLSC equation with little difficulty.[12] In the case of right-running acoustic waves, and retaining x and t as the independent variables, this celebrated PDE assumes the form [2,7]

$$2u_t + 2(1 + \epsilon\beta u)u_x = (\mathrm{Re_d})^{-1}u_{xx}. \tag{A.1}$$

Seeking travelling wave solutions, we set $u(x, t) = \mathfrak{F}(\xi(v_{\mathrm{Beq}}))$ in Eq. (A.1) and then impose the asymptotic conditions $u \to 1, 0$ as $\xi \to \mp\infty$. On solving the resulting Bernoulli equation subject to the wavefront condition $\mathfrak{F}(0) = 1/2$, we obtain the well known result

$$\mathfrak{F}(\xi) = \frac{1}{2}\left\{1 - \tanh[2\xi(v_{\mathrm{Beq}})/\ell_{\mathrm{Beq}}]\right\}, \tag{A.2}$$

where the wave speed and the shock thickness are respectively given by $v_{\mathrm{Beq}} := 1 + \epsilon\beta/2$ and $\ell_{\mathrm{Beq}} = \ell_1$. Here, we note that $v_1 > v_{\mathrm{Beq}}$, a fact easily established with the aid of Eq. (18), and that, apart from the difference in the wave speeds, $\mathfrak{F} = F$ for $n = 0, 1$; i.e., the plot of $\mathfrak{F}$ vs. ξ is *identical* to the F vs. ξ solution curves corresponding to Eqs. (11) and (12).

Below in Table A.1, for the same values of β used to generate Tables 1–3 and Figs. 3 and 4, the Burgers' TWS is compared with its exact counterpart (Eq. (43)), using once again the three metrics defined in Eqs. (48).

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[12] Actually, if one begins instead with Kuznetsov's equation, then fewer steps are required in Crighton's [7] derivation.